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Bargaining and Rent Seeking: Asymmetric Equilibria with Pure Investment Strategies

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Bargaining and Rent Seeking: Asymmetric Equilibria with Pure Investment Strategies

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Abstract

We consider a Baron-Ferejon type bargaining model with recognition probabilities determined by a Tullock contest. The contest is carried out once-and-for-all before bargaining a la Yildirim (2010). It is known that for ex-ante symmetric players, there do not exist symmetric stationary subgame perfect equilibria (SSPE) in pure investment strategies. In this paper, we show the existence of an asymmetric SSPE if players are sufficiently patient. In these equilibria, players are divided into large vs. moderate investors, where the latter group size equals the majority quota.

Key words: bargaining, rent seeking, subgame perfection
JEL -code: C72, C78, D70

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1 Introduction

The Baron-Ferejohn model of legislative bargaining (Baron and Ferejohn (1989)) received significant attention in the literature as a natural extension of the sequential bargaining model. Especially the finding of the uniqueness of the stationary subgame perfect equilibrium (SSPE) payoffs under general recognition probabilities (Eraslan (2002)) inspired the attempts to extend and apply this model in several respects. (See Eraslan and Evdokimov (2019), for example.) Yildirim (2007, 2010) has endogenized recognition as a Tullock contest (Tullock (1980), see also Ali (2015) for an alternative formulation as an all-pay auction). Such a model would give a potential explanation of the bargaining power and how an institutional arrangement might affect the consequence of bargaining as well as the use of resources by the legislators. Yildirim (2007) analyzed mainly the case where investments are made in the beginning of every bargaining period ("transitory recognition") while in the same paper, another version of investment (persistent recognition) is introduced too, where investment is made once and for all at the beginning of the negotiation. With the persistent recognition, the SSPE payoffs remain unique in pure strategy efforts when the voting rule is unanimity, which is examined in Yildirim (2010) for instance. However, the characterization of equilibrium has been elusive when the rule is non-unanimity, as previously noted in Querou and Soubeyran (2011) for example. (Here, we deal with the case where the issue is purely distributive. The story is different if the issue is represented by a single-dimensional policy choice. see Cardona and Polanski (2011))

The remainder of this paper is organized as follows. Section 2 presents the model and reviews known results about equilibria in the bargaining game. Section 3 presents asymmetric equilibria with pure investment strategies, and Section 4 concludes.

2 The model

We analyze a two-stage game in which players first make investments to improve their position in the bargaining game played in the second stage. Prior to bargaining, the first-stage investments become common knowledge. This paper focuses on the equilibrium strategies of the first stage. The results of the second stage are well known. Therefore we would like to state some of the known results for the second stage without a proof.

2.1 The bargaining game

There are n players indexed by $i \in N = \{1, \ldots, n\}$ who can divide the cake of size 1 if an agreement $x = (x_1, \ldots, x_n)$ is reached on how to divide the cake when at least q players agree where $n/2 < q < n$. (The case with $n = q$ is the pure bargaining or the unanimity rule and we have sufficient understanding of this situation (e.g. Yildirim (2010)). Player i's utility from an agreement x is simply $x_i, i = 1, \ldots, n$, the size of the player's share. If no agreement is reached, each player obtains utility 0. Agreements x must then satisfy $x_i \geq 0$ (individual rationality) and $\sum_i x_i = 1$ (efficiency). Time periods are indexed by natural numbers $t \in \mathbb{N} = \{0, 1, \ldots\}$.

This game starts in period 0 wherein a *proposer* $i \in N$ is selected. Each player i has a probability $p_i \geq 0$ of being chosen. The selected player i offers an agreement x^i, and the other players, the *responders*, announce Yes or No. In the literature, both simultaneous and sequential voting rules are studied. However, we describe the game with simultaneous voting rule for simplicity and remark on the case of sequential voting later. (With suitable refinement respectively, both yield the identical set of equilibrium payoffs.) If more than or equal to q players say Yes, the game ends with an agreement x^i. If not, the game then moves to the next period and the proposer is selected randomly with probabilities $p_i, i = 1, \ldots, n$. If an agreement x is reached in period t, then player $i's$ utility is $\delta^t x_i, i \in N$, where δ is the common discount factor, $0 < \delta < 1$.

We assume that in each period t, all past decisions of all players are common knowledge. Then if no agreement has been reached before period t, a new selection of the proposer begins at $t + 1$.

A *history* is any sequence of actions taken in the game (including chance moves where the nature chooses the proposer for that period if no agreement was reached in the previous period or at the beginning of the second stage). If an agreement is reached in the history, then it must terminate there. A player $i's$ *strategy* s_i of is a plan specifying for each history, (1) what is the proposal to be proposed when the last component of the history specifies i as the chosen proposer, and (2) if the last component of the history is a proposal made by the other player, whether this player votes Yes or No is specified. (In the case of sequential voting, the last component may be another player's response.)

A profile $s = (s_1, \ldots, s_n)$ is a Nash equilibrium if s_i maximizes player i's utility when i believes that other players j choose s_j. A subgame is a game beginning after a history where the last action(s) in the history is a common knowledge among players. A Nash equilibrium s is *subgame perfect*, if it is a Nash equilibrium in every subgame of a bargaining game. A strategy is *stationary*, if a strategy prescribes the same action for any history if the actions within a period are identical. (For infinite games, stationarity could be define by the property that in the identical subgames the same subgame strategy is induced. See Osborne (2023)). A subgame perfect equilibrium s is stationary if every strategy s_i is stationary. Kalandrakis (2015) utilized the stage undominated (Baron and Kalai (1993)) notion to refine SSPE in the simultaneous voting protocol (e.g. Osborne (2023)). This refinement yields set of payoffs identical to that given by the SSPE under the sequential voting protocol as characterized by Eraslan (2002).

Under sequential voting, upon a proposal, within a period, responders announce Yes or No sequentially given a certain ordering on N (several variants on this may exist (cf. Eraslan and McLennan (2013)). If $q-1$ players say Yes the game ends. If the quota is not reached within that period, the game moves into the next period. All the responses are known by other players immediately, and the definition of strategies, subgames, and stationarity are modified to accommodate these changes.

Eraslan (2002) (Theorem 5) demonstrates that the SSPE payoff is unique and for each player, this payoff is non-decreasing in the recognition probability with a common discount factor (Corollary 1). Kalandrakis (2015) reported a characterization of this set of equilibrium payoffs considering that the set of equilibrium payoffs of SSPE which is stage undominated coincides with the set of SSPE outcomes of the game under the sequential voting (theorem 1, 2 and Footnote 6). Eraslan and McLennan (2013) also describes identical outcomes in terms of the "reduced equilibrium" in a more general setup.

2.2 Second Stage Equilibrium

Let $p = (p_1, \ldots, p_n)$ be a vector of recognition probabilities. Given p, and with the assumption of the identical discount factor, we shall describe the expressions for SSPE payoffs, given by Eraslan (2002) and Kalandrakis (2015) (where distinct discount factors are allowed).

By the stationarity, we could represent the recursive relationship among equilibrium payoffs. Let player i's continuation value be v_i. A key player is player q with the q-th smallest recognition probability and set of players with the same continuation payoff as player q. Define the set $M(p, \delta) = \{j : v_j = v_q\}$, $L(p, \delta) = \{j : v_j < v_q\}$, and $H(p, \delta) = \{j : v_j > v_q\}$.(This definition differs from those of Kalandrakis (2015) only slightly which we comment on later.) Where it is convenient, we omit p and δ. We refer to $\{L, M, H\}$ as the equilibrium partition, which would vary with profiles of recognition probabilities. Moreover, M is nonempty while L and/or H could be empty.

For the strategy to be optimal, a proposer i should make offers yielding a minimal amount to other players who would vote for the proposal δv_j i.e. to those members of the minimal winning coalition W including the proposer which minimizes $\sum_{j \in W, j \neq i} \delta v_j$. In a SSPE, this offer is accepted by the members.

In Kalandrakis (2015), equilibrium payoffs given p and an equilibrium partition are characterized by two variables. One is the "marginal reservation value" $r = \delta v_q$, the largest amount paid to a single coalition member in the proposal wherein if the proposer is in H, r can be considered as the opportunity cost. The other is the "maximal proposer surplus" S, which represent the gain from being a proposer, i.e. $S = 1 - \delta \sum_{j \in L} v_j - (q - |L|)r$ wherein if the proposer is in H, opportunity cost r is subtracted. Combining the conditions that $\sum_{j \in N} v_j = 1$, that is the sum of v_i must be equal to 1, Kalandrakis (2015) obtain a system of linear equations in S and r.

Lemma 1 of Kalandrakis (2015) indicates that $v_i = p_i(S+r)$ if $i \in H$ and $v_i = \frac{p_i S}{1-\delta}$ if $i \in L$. The difference in the definition of M and H between this study and that of Kalandrakis is that we include i in M, if $v_i = p_i(S+r) = \frac{r}{\delta}$ rather than in H.

Write $\sum_{j \in T} p_j = p(T)$ for any $T \subset N$.

$$v_i = \begin{cases} p_i\{|M| - \delta(q - |L| - p(H))\}/D & i \in L \\ (1-\delta)p(M)/D & i \in M \\ p_i(1-\delta)\{|M| - \delta(q - |L| - p(H) - p(M))\}/D & i \in H \end{cases} \quad (1)$$

where

$$D = (\delta p(H) + |M|)\big[1 - \delta + \delta p(L)\big] - \delta(q - |L|)\big[(1-\delta)p(H) + p(L)\big]. \quad (2)$$

(These expressions are employed in Imai and Salonen (2012). To obtain this, solve the simultaneous equations in S and r described above, which yield $S = \delta(1-\delta)[|L| + \frac{|M|}{\delta} + p(H) - q]/D$ and $r = \delta(1-\delta)p(M)/D$.)

Moreover, to confirm the equilibrium properties, determining whether the partition is an equilibrium partition is necessary. Lemma 1 of Kalandrakis (2015) reports the conditions that $i \in H$ if $\frac{\delta p_i}{1-\delta p_i} S > r$, and $i \in L$ if $\frac{\delta p_i}{1-\delta} S < r$ (modified for the definition of M and H here). Quantity $R_i = \frac{r - \delta p_i(r+S)}{\delta(1-p_i)r}$ enables us to obtain one interpretation of these conditions. Provided that $r > 0$, $R_i \geq 0$ if $r \geq \frac{\delta p_i}{1-\delta p_i} S$, and $R_i \leq 1$ if $r \leq \frac{\delta p_i}{1-\delta} S$. The recursive relationship for the continuation value of a player in M can be written in the form of $\frac{r}{\delta} = p_i(r+S) + R_i(1-p_i)r$. R_i is the probability of being invited into a majority coalition when $i \in M$ is not a proposer. If $i \in H$, i is not invited, and if $i \in L$, i is always invited. Hence, the above condition represents the condition so that i's payoff can be probabilistically adjusted to attain the level r/δ.(In Eraslan (2002), the basic relations are expressed using the probabilities of each player's inviting other players by proposing the discounted continuation value in the proposal.)

Based on these properties, we state two Lemmas.

Lemma 1: If $\frac{1}{n-\delta q+\delta} \geq p_i \geq \frac{1-\delta}{n-\delta q}$ holds for each i then the equilibrium partition is given by $M = N$.

Proof: If at an equilibrium, $M = N$, then $S = \frac{n-\delta q}{n}$ and $r = \frac{\delta}{n}$. Then for each i, $\frac{\delta p_i}{1-\delta p_i} \geq \frac{r}{S} \geq \frac{\delta p_i}{1-\delta}$ holds; hence the equilibrium condition is satisfied. □

(Note that $\frac{1}{n-\delta q+\delta} \geq \frac{1}{n} \geq \frac{1-\delta}{n-\delta q}$ holds.)

Lemma 2: Given $p \gg 0$, there exists $\delta' < 1$ such that for any $\delta > \delta'$, $L(p,\delta) = \emptyset$.

Proof: Suppose the claim does not hold. Then for any $\delta' < 1$, there exists $\delta > \delta'$ so that $L(p,\delta) \neq \emptyset$. For $i \in L(p,\delta)$, $p_i\{|M(p,\delta)| - \delta(q - |L(p,\delta)| - p(H(p,\delta))\}/D < (1-\delta)p(M(p,\delta))/D$ must hold by the definition of L and M. Since $D > 0$, this becomes $p_i\{|M(p,\delta)| - \delta(q - |L(p,\delta)| - p(H(p,\delta))\} < (1-\delta)p(M(p,\delta))$.

Note that if $q = |L(p,\delta)| + |M(p,\delta)|$, then $p(H(p,\delta)) > \frac{n-q}{n} > 0$ and if $H(p,\delta) = \emptyset$, then $1 \leq |L(p,\delta)| + |M(p,\delta)| = q$ holds. Thus given $p \gg 0$, $\{|M(p,\delta)| - \delta(q - |L(p,\delta)| - p(H(p,\delta))\} > \max\{\delta(|M(p,\delta)| + |L(p,\delta)| - q), \delta p(H(p,\delta))\} > \delta(\frac{n-q}{n}) > 0$ holds for any δ, on the one hand. On the other hand, there exists $\delta" < 1$ such that $(1-\delta")\min p_j < \frac{1}{2n}$. These two inequalities cannot hold at the same time. □

(Note that this relation holds when δ changes given $p \gg 0$. If p changes given $\delta < 1$, then for smaller values of p_i, $L(p,\delta)$ becomes non-empty.)

2.3 Investment Game

Before the bargaining game begins, players can make investments that increase their recognition probabilities. Let $e_i \geq 0$ denote the amount (money, effort *etc.*) invested by player i. We assume that p_i depends on the investments $e = (e_1, \ldots, e_n)$ in the following simple way:

$$p_i(e) = \frac{e_i}{\sum_{j \in N} e_j}, \ i \in N \tag{3}$$

If no one invests, each player has the same probability $1/n$ of being selected as the proposer.

We assume that players' investment costs are linear and identical: investing an amount e_i costs ce_i, where $0 < c$ for all $i \in N$.

A player's expected payoff $U_i(e, \delta)$ from investment decisions e in the whole game (investment and the bargaining stage) is given by

$$U_i(e, \delta) = u_i(e|\delta) - c_i e_i, \; i \in N, \tag{4}$$

where $u_i(e|\delta)$ is given by v_i in (1).

We seek for *Nash equilibrium* of the first stage game where payoff is given by $(U_i)_{i \in N}$, and assume that investment levels become common knowledge before the bargaining game begins. It is known that there is no symmetric pure strategy equilibrium in this symmetric game (e.g. Querou and Soubeyran (2011)).

3 Asymmetric equilibrium with pure investment strategies

While there are no equilibria in which all players make the same investment, there are equilibria in which players (non-randomly) choose different levels of investment exist if the discount factor is sufficiently large and the investment costs are identical. First we show candidate equilibrium.

As has been mentioned, there is no symmetric equilibrium with pure investment strategy, and we see this from Lemma 1 that if each player invests the same level, then $M = N$. Thus each player's bargaining payoff is $1/n$; and there is an incentive to reduce investment (if investment is positive, and if the investment level is 0, there is an incentive to increase the investment level). Lemma 1 also states that even if investment levels differ, thus the same argument holds if $M = N$. Thus, one must expect that if there are equilibria in pure investment strategies, some players will be in L or H.

Although Lemma 2 does not imply the non-existence of an equilibrium with non-empty L, it indicates that such an equilibrium would involve very small probabilities and the corresponding payoffs are difficult to analyze. Therefore, we can hope to find an equilibrium with a partition involving only M and H. In particular, the case with $|M| = q$ and $|H| = n - q$ would be atractive. Since M is a minimal winning coalition, a player in H cannot receive a positive bargaining payoff unless that player becomes the proposer. Such equilibria exist and can be derived from interior first-order conditions.

First, we briefly describe the candidate equilibria and their properties, before establishing that they are indeed equilibria.

Let $e(T) = \sum_{i \in T} e_i,$ for any $T \subset N$.

Given $|M| = q$ and $|H| = n - q$, the bargaining payoffs and S, r become

$v_i = \frac{p(M)}{q-\delta(q-1)p(H)} = \frac{e(M)}{qe(M)+Qe(H)}$ for $i \in M$

$v_i = \frac{Qp_i}{q-\delta(q-1)p(H)} = \frac{Qe_i}{qe(M)+Qe(H)}$ for $i \in H$

$S = \frac{q(1-\delta)+\delta p(H)}{q-\delta(q-1)p(H)} = \frac{q(1-\delta)e(N)+\delta e(H)}{qe(M)+Qe(H)}$

$r = \frac{1-S}{q} = \frac{\delta e(M)}{qe(M)+Qe(H)}$

($Q = (1-\delta)q + \delta$. $Q > 1$ and $Q \downarrow 1$ as δ tends to 1.)

Thus, the ratio between S and r is approximately equal to that between $e(H)$ and $e(M)$ when δ is sufficiently large. Moreover, recall that the bargaining payoff of a player in M depends on that player's own investment level only through $e(M)$. We found that this is reflected in the formula so that the benefit of investment by a player in M is shared equally by the other members.

Next, the marginal benefits of investment represented by the bargaining payoffs are as follows.

$\frac{\partial v_i}{\partial e_i} = \frac{Qe(H)}{(qe(M)+Qe(H))^2}$ for $i \in M$

$\frac{\partial v_i}{\partial e_i} = \frac{Qqe(M)+Q^2(e(H)-e_i)}{(qe(M)+Qe(H))^2}$ for $i \in H$.

These are just results from differentiating fractions but they give a useful relationship. At an equilibrium satisfying first-order conditions, these two must be equal to c, so we have

$qe(M) - Qe_i + (Q-1)e(H) = 0$.

Since the same first-order condition holds for each i in H,

$$e_i^\delta = Q'qe^\delta(M) \tag{5}$$

where

$$Q' = \frac{1}{Q-(n-q)(Q-1)} \tag{6}$$

($Q' = 1$ when $n - q = 1$ and otherwise $Q' > 1$ for δ close to 1. $Q' \downarrow 1$ as δ tends to 1.)

Finally we obtain

$$e_i^\delta \; for \; i \in M \text{ satisfies } \sum_{i \in M} e_i^\delta = e^\delta(M) = \frac{(n-q)QQ'}{q(1+(n-q)QQ')^2c} \tag{7}$$

$$e_i^\delta = \frac{(n-q)QQ'^2}{(1+(n-q)QQ')^2c} \; for \; i \in H \tag{8}$$

The condition

$$e_i^\delta \geq \max\{4(1-\delta)\frac{q(n-q)}{(n-q+1)^2c}, U_i^{-1}(u_i(e_i^{\delta 0}, e_{-i}^\delta|\delta))\} \tag{9}$$

must be satisfied for $i \in M$. We have
$v_i = \frac{1}{q(1+(n-q)Q'Q)}$ for $i \in M$
$v_i = \frac{QQ'}{(1+(n-q)Q'Q)}$ for $i \in H$
$S = \frac{(1-\delta)+(n-q)QQ'}{(1+(n-q)QQ')}$
$r = \frac{1-S}{q} = \frac{\delta}{q(1+(n-q)QQ')}$.

Observe that the investment level of each player in H is greater than or equal to q times the aggregate investment level of players in M.

The net payoffs in this equilibrium are

$$U_i^\delta = \frac{1}{q(1+(n-q)Q'Q)} - \frac{(n-q)QQ'e_i^\delta}{q(1+(n-q)QQ')^2 e^\delta(M)} \tag{10}$$

$$= \frac{(1+(n-q)QQ')e^\delta(M) - (n-q)QQ'e_i^\delta}{q(1+(n-q)QQ')^2 e^\delta(M)} \text{ for } i \in M \tag{11}$$

$$U_i^\delta = \frac{QQ'}{(1+(n-q)Q'Q)} - \frac{(n-q)QQ'^2}{(1+(n-q)QQ')^2} \tag{12}$$

$$= \frac{QQ'[(1+(n-q)QQ' - (n-q)Q']}{(1+(n-q)QQ')^2} \text{ for } i \in H. \tag{13}$$

We will show that e^δ with an additional constraint on e_i^δ $(j \in M)$ is an equilibrium of the reduced game when δ is close to 1.

We need to verify that the above strategies are the best responses to each other, and to do this, one needs to examine all the equilibrium partitions that emerge when a player changes the investment level with a payoff function for each partition. In particular, a partition with non-empty L would emerge when a player reduces the investment level, and there is another possibility for L to emerge depending on the balance of investment levels among other players as well. To illustrate, we present the case for three-person games using a diagram to show how the way partitions vary. Then we give the main result and give some discussion.

Example 1. If $N = \{1,2,3\}$, $q = 2$ and $c > 0$ for every $i \in N$, then let the investment level of player 3 be $e_3 = (2-\delta)/c(3-\delta)^2$, and let the investment levels of players 1 and 2 satisfy $e_1 + e_2 = (2-\delta)/2c(3-\delta)^2$ with $0 < e_1/e_2 <$

1/5. Then there exists a $\bar{\delta} < 1$ such that for $\delta > \bar{\delta}$, there exists an $SSPE$ with investment levels $e = (e_1, e_2, e_3)$. Evaluating at the limit as δ tends to 1, the equilibrium recognition probability p_3 of player 3 is 2/3. The recognition probabilities of players 1 and 2 satisfy $p_1 + p_2 = 1/3$.

Figure 1 shows an example of an equilibrium for the case $n = 3$. The triangle represents the probability simplex. The solid curves delineate each region corresponding to an equilibrium partition. For clarity, the membership is written for only some of them. It can be seen that the center of the triangle, corresponding to the equal investment levels, is in the middle of the region for $M = N$. Thus the bargaining payoff is constant at 1/3. Because of the positive investment cost, this point cannot be an equilibrium, as already argued.

The thick line segment represents one set of equilibria. The dashed lines represent the locus of the probability profiles when one player deviates and the point where three lines intersect is an example equilibrium for this illustration. Observe that as the investment level of each player decreases, that player may be a single member of L. Moreover as player 2's investment level varies, player 1 may approach the region wherein 1 becomes a member of L in the middle range of player 2's investment (or probability) level.

Note that in an equilibrium one large investor (player 3) and two moderate investors (players 2 and 1) coexist, with 2 investing more than 1. (With more than 3 players, the number of large investors could be higher.)

A special feature of the three-players case is the property that players 1 and 2 must invest different amounts in an equilibrium. If the large investor, player 3, reduces the investment level, then all players become M players to receive the same bargaining payoff, 1/3. To maintain the the large investor's incentive for investment, in an equilibrium, players 1 and 2 must create a gap between their investment levels. Consequently that a reduction in the investment level by player 3 induces player 2 to become an H player, and player 3 cannot enjoy the payoff, 1/3.However, this is not necessarily the case for $n \geq 4$, since a payoff level of 1/4 or less is not attractive for a large investor. Another peculiarity of the case $q = n - 1$ is the fact that after changing the investment level, all players can become moderate investors, as mentioned above. But if $n - q > 1$, there is another large investor exists, so an H player would not disappear by unilaterally reducing the investment level of a single large investor.

A side observation is that regions corresponding to partitions containing non-empty L are located relatively close to the boundary of the probability simplex (as was suggested by Lemma 2). We observe that partitions containing a non-empty L shrink to the boundary as the discount factor tends to 1 in this case. This is because no matter how small a recognition probability is, as long as it is positive, a sufficiently large discount factor makes this player quite expensive so that this player can never be invited into a majority coalition. In general this property holds, which makes our analysis much simpler since the behavior of the payoffs of players in L is more complicated. Of course, if the investment level is 0, this argument does not hold in general. Thus, the bargaining payoff could be discontinuous at the boundary in the limit.

What we claim in this paper is;

Proposition 1. *There is an asymmetric investment equilibrium when* δ *is close to* 1*.*

Proof. The proof is in the Appendix, where the candidate equilibria with more restrictions are shown to be equilibria when the discount factor is large. The main reason we need a high discount factor is to show that there is no best response when a player belongs to L, with a very low investment level. □

Remark 1*.* Since M players receive the same bargaining payoff level at the bargaining stage, the more they invest, the lower their net payoff. However, the M player's best ex-ante net payoff remains below that of an H player.

Let $i \in M$ and let the equilibrium payoff of a player in H be U_H^δ.

$$\begin{aligned} U_i^\delta &= \frac{(1+(n-q)QQ')e^\delta(M) - (n-q)QQ'e_i^\delta}{q(1+(n-q)QQ')^2 e^\delta(M)} \\ &< \frac{(1+(n-q)QQ')}{q(1+(n-q)QQ')^2} \\ &\leq \frac{QQ'[(1+(n-q)QQ' - (n-q)Q']}{(1+(n-q)QQ')^2} \\ &= U_H^\delta. \end{aligned}$$

Remark 2*.* As observed above, the game defined by the limit equilibrium payoffs of the bargaining game when δ tends to 1could be considered. For any strictly positive p, there exists δ' such that for $\delta > \delta'$, L is empty by

Lemna 2 and the bargaining equilibrium partition is constant. Thus limit payoffs exist.

Specifically, for p with some $p_i = 0$, the limit also exists, but the limit payoffs there may not be continuous in p. Thus, for a player in M, 0 would not be an equilibrium investment level. Equilibria defined by the first-order conditions above can be defined analogously. The limit equilibria and payoffs are

$$\begin{aligned} e^1(M) &= \frac{(n-q)}{[(n-q)+1]^2 c}, \quad \text{for} \quad i \in M, \\ &\quad \text{where} \quad e^1(M) = \sum_{i \in M} e_i^1, \quad \text{and} \quad e_i^1 > 0, \\ e_i^1 &= q e^1(M), \quad \text{for} \quad i \in H, \end{aligned}$$

and

$$\begin{aligned} U_i^1 &= \frac{e^1(M) + [(n-q)](e^1(M) - e_i^1)}{q\{(n-q)+1\}^2 e^1(M)}, \quad \text{for} \quad i \in M \\ U_i^1 &= \frac{1}{\{(n-q)+1\}^2}, \quad \text{for } i \in H. \end{aligned}$$

Using the limit values, analysis such as comparative statics could become more transparent.

Social cost of rent-seeking in the limit is

$$\frac{(q(n-q)+1)(n-q)}{q[(n-q)+1]^2},$$

and its derivative with respect to q is

$$\begin{aligned} & \frac{q[(n-q)+1]\{(n-2q)(n-q) - (q(n-q)+1)\}}{q^2[(n-q)+1]^3} \\ & - \frac{(q(n-q)+1)(n-q)\{(n-q+1) - 2q\}}{q^2[(n-q)+1]^3} \\ & = \frac{-n - (n-q)(2q(q-1)+n)}{q^2[(n-q)+1]^3} < 0. \end{aligned}$$

This result is similar to that obtained in Yildirim (2007) where the unanimity rule is shown to minimize the social cost.

4 Concluding remarks

We have shown that asymmetric equilibria with pure investment strategies for the game to influence the bargaining position in legislative bargaining exist under the assumption of identical costs and identical and large discount factors. In these equilibria, players are divided into two groups: one with a very high level of investment and higher payoffs but no chance of being invited into a majority coalition proposed by other players, and the other with a relatively moderate level of investment, the same bargaining payoff within this group and non-negative probability of being invited. Thus, multiplicity exists in stationary equilibria of two-stage games with substantial payoff inequality.

Within the group of moderately investing players, there is free rider benefit for those players with lower investment levels (if there is a difference in investment levels within this group). Basically, this is a consequence of stationary equilibrium which requires that the bargaining payoffs equalize as the probability of being invited into a majority coalition adjusts. Another important driving force is identical and linear costs so that players' investment levels can be interchanged to create space for different profiles of investment levels. Therefore, extending the analysis to heterogeneous investment costs would be a worthwhile future research direction. However, from the equilibria obtained here, one could provide insights into whatmight happen in mixed strategy equilibria in the pure distributive and majoritarian bargaining game.

Future works remains to tesyt the possibility of the pure strategy equilibria with other patterns remain, especially if there are equilibria with other cardinalities of $|M|$ and $|H|$, or if there is an equilibrium with a non-empty L. Moreover, it would be interesting to introduce a more general form of the context success functions.

APPENDIX

This proposition is proved by explicitly checking the equilibrium conditions for the candidate equilibria that is given in (4).

In the following, we first state the criteria for judging an equilibrium partition and we show the partitions that arise when a player deviates from the candidate equilibria. After showing the payoff functions when each player deviates unilaterally, we state the choice of discount factors. Finally, we confirm the best response properties. Recall that a *bargaining payoff* is an equilibrium payoff of the bargaining game and a *net payoff* means a payoff of the investment game.

A.1. Equilibrium Partitions

From the characterization of equilibrium payoffs in the bargaining stage in Section 2, for each p, there corresponds an equilibrium partition, $\Pi(p,\delta) = \{L(p,\delta), M(p,\delta), H(p,\delta)\}$, (we can omit an empty set). Given p, $M(p,\delta)$ is uniquely determined as the set of players who receive the same bargaining payoff level as player q, and hence $H(p,\delta)$ and $L(p,\delta)$ are also uniquely determined. In the following, we will mostly write H, M, L, since we consider only the equilibrium partition associated with each p, or e.

Recall that in a bargaining equilibrium, for $i \in M, 0 \leq R_i \leq 1$ must hold, and thus we have the following.

$$(1-\delta)p(M) \leq p_i(|M| - \delta q + \delta|L| + \delta p(H)) \tag{14}$$

and

$$\underset{i\in M(p)}{max}\, p_i \leq \frac{p(M)}{\delta(p(M)+p(H)) + |M| + \delta|L| - \delta q} \tag{15}$$

With $L = \emptyset$, the right-hand-side of (5) becomes $\frac{p(M)}{|M|+1-q}$ and at the candidate equilibrium, this condition is almost automatically satisfied for M with $|M| = q$.

Given $e_{-i} \gg 0$, define $e_i^{\delta 0} = \sup\{e_i : i \in L(p(e_i, e_{-i}), \delta)\}$. ($e_i^{\delta 0} > 0$.) For $e_i^{\delta 0}$, $r = \frac{Sp_i}{1-\delta}$ holds.

Observation 1: Fixing a partition, e_i defined by $r = \frac{Sp_i}{1-\delta}$ tends to 0 as $\delta \longrightarrow 1$. Since there are finitely many possible partitions. $e_i^{\delta 0}$ tends to 0 as $\delta \longrightarrow 1$.

It is also necessary to consider the possibility that if the investment level of another player varies, a small investor can become an L player (as shown in Example 1). For our purpose, assume a non-zero investment profile $e \gg 0$. Moreover, assume that the corresponding equilibrium partition is with an empty L. Thus, the smallest investor belongs to M. Pick a player i and let e_i vary. Then let $\rho(e_i, e_{-i}, \delta)$ be the critical investment level of $j \in \arg\min\{p_{j'} : j' \in M\backslash\{i\}\}$, to stay in M rather than in L.

$$\rho(e_i, e_{-i}, \delta) = \frac{(1-\delta)p(M)}{(|M| - \delta q + \delta|L| + \delta p(H))} \tag{16}$$

(Note that the corresponding equilibrium partition would change as e_i changes. Later, we explicitly compute ρ when $L = \emptyset$.)

When $\{i\} \in L$:

$u_i(e_i, e_{-i}|\delta) =$

$$\frac{[(|M| - \delta q + \delta|L|)e(N) - e(H)]e_i}{(e(H) + |M|e(N))((1-\delta)e(N) + \delta\frac{e(L)}{e(N)}) - \delta(q - |H|)[(1-\delta)e(H) + e(L)]} \tag{17}$$

We know that the above function is continuous (given a partition), non-decreasing in e_i and is equal to 0 only when $e_i = 0$.

For any investment profile e, let $e(T)$, with $T \subset N$, represent the sum $\sum_{j \in T} e_j$.

Lemma 3: The candidate equilibrium profiles are consistent with the proposed partition, if $\min e_i^\delta > 4(1-\delta)\frac{(n-q)q}{[(n-q)+1]^2 c}$ and δ satisfies $\rho(e^\delta, \delta) < 2(1-\delta)\frac{(n-q)q}{[(n-q)+1]^2 c}$.

Proof: The way we prove this lemma is to first check the consistency for a partition with $L = \emptyset$ and then check that the condition for no player to be in L is indeed satisfied. $n - q$ players in $H(p^\delta)$ invest at the same level. Therefore they are all either in M or in H. However, $|M(p^\delta)| \geq q$. So, the possible partition with $L = \emptyset$ is with $M(p^\delta) = M^\delta$ or $= N$. If the latter is the case, $R_i \geq 0$ must hold and so $\underset{i \in N}{max} p_i \leq \frac{1}{\delta + n - \delta q}$ while $\underset{i \in N}{max} p_i = \frac{qQ'}{(n-q)qQ'+1} > \frac{1}{\delta + n - \delta q}$ for δ close to 1, which is a contradiction.

Finally, for any player not to be in L , we must have $\min e_i^\delta > \rho(e^\delta, \delta) = \frac{(1-\delta)e^\delta(M)}{(1-\delta)q + \delta\frac{(n-q)qQ'}{(n-q)qQ'+1}}$ where ρ is computed based on the candidate equilibrium partition. Since $e^\delta(M^\delta) = \frac{Q(n-q)q}{[Q(n-q)+1]^2 c}$ and in the limit as δ tends to 1, this value becomes $\frac{(n-q)q}{[(n-q)+1]^2 c}$. For δ sufficiently large, we have

$$\rho(e^\delta, \delta) = \frac{(1-\delta)e^\delta(M)}{(1-\delta)q + \delta\frac{(n-q)qQ'}{(n-q)qQ'+1}} < 2(1-\delta)e^\delta(M) < 4(1-\delta)\frac{(n-q)q}{[(n-q)+1]^2 c} < \min e_i^\delta$$

.

Therefore the candidate partition is indeed the equilibrium partition associated with e^δ. □

A.2. Deviation

To confirm the best response property, we need to examine each player's bargaining equilibrium payoffs when each player deviates unilaterally, i.e. $u_i(e_i, e_{-i}|\delta)$ for each i. To this this, we need to identify the equilibrium partitions that would vary with a deviation. Two observations help to understand the patterns.

Observation 2: Since players in H^δ choose the same investment levels in the candidate equilibria, except for the deviating player, the remaining players must be in the same component of the partition.

Observation 3: Below $e_i^{\delta 0}$, a further reduction in investment level of player i' does not change the partition.

Below, we list the switching levels and partitions for $e_i > e_i^{\delta 0}$, for $i \in M^\delta$ and H^δ, separately. Figure 2 shows the payoff of player i in M^δ is depicted. Point A represents the equilibrium investment level and the equilibrium bargaining payoff for i. The line passing through A is an iso-net payoff line with a slope c. Moreover, the line starting from point B has the same slope, c. To the left of point C, player i becomes an L player.

Figure 3 shows the payoff of a player in H^δ. Point A represents the equilibrium investment level and the bargaining payoff for i. The line passing through A is an iso-net payoff line with a slope c. In addition, the line starting at point B is the upper support of the graph to the left of point B. At point C, the partition switches and the bargaining payoff is less than in the case where the switch does not occur. (This does not happen if there are more than or equal to the two largest investors in M^δ. Also, the case $n - q = 1$ is different, because there is only one player in $H(p^\delta)$, when this player reduces the investment level, $M = N$ emerges. Thus, the bargaining payoff remains constant for this partition.) After showing the payoffs, we examine the possibility of other players becoming L players, we shall check ρ defined above, and we confirm that no such case arises.

Let $V_{T,s}^\delta$ be the function representing the bargaining payoff of a player belonging to the set T ($= M$ or H) with the cardinality s, given $L = \emptyset$. Below, we show the bargaining payoffs corresponding to each player's deviation, with the equilibrium partition as described above. (The case with non-empty L is omitted.) We summarize the payoff functions and the critical value of the investment level at which a small investor becomes an L player.

We define

$f_{1i}^\delta(e_i) = \frac{e^\delta(M) - e_i^\delta + e_i}{q(e^\delta(M) - e_i^\delta + e_i) + Qe^\delta(H)}$

$f_2^\delta(e_i) = 1/n$

$f_{3i}^\delta(e_i) = \frac{(n-1-\delta(q-1))e_i}{(n-1-\delta(q-1))e_i + (n-1)(e^\delta(H) + e^\delta(M) - e_i^\delta)}$

$g_1^\delta(e_i) = \frac{e^\delta(M) - e_q^\delta + e_i}{q(e^\delta(M) - e_q^\delta + e_i) + Q(e^\delta(H) - e_i^\delta + e_q^\delta)}$

$g_2^\delta(e_i) = \frac{e^\delta(M) + e_i}{(q+1)(e^\delta(M) + e_i) + (Q+1)(e^\delta(H) - e_i^\delta)}$

$g_3^\delta(e_i) = \frac{Qe_i}{qe^\delta(M) + Q(e^\delta(H) - e_i^\delta + e_i)}$

Note that all these functions are defined on $\mathbb{R}_+$.When $i \in M^\delta$ deviates,

	$e_i^{\delta 0} \leq e_i < \underline{e}_i^\delta$	$\underline{e}_i^\delta \leq e_i \leq \overline{e}_i^\delta$	$\overline{e}_i^\delta < e_i$
Π	$i \in M^\delta$	$i \in M = N$	$\{i\} = H$
u_i	$V_{M,q}^\delta(e_i, e_{-i}^\delta) = f_{1i}^\delta(e_i)$	$V_{M,n}(e_i) = f_2^\delta(e_i) = 1/n$	$V_{L,1}^\delta(e_i, e_{-i}^\delta) = f_{3i}^\delta(e_i)$
u_i'	$+$	0	$+$
u_i"	$-$	0	$-$
ρ	$\frac{(1-\delta)e(M)e(N)}{(1-\delta)qe(N)+\delta e(H)}$	$\frac{(1-\delta)e(N)}{(n-\delta q)}$	$\frac{(1-\delta)e(M)}{n-1-\delta q+\delta \frac{e_i}{e(N)}}$
ρ'	+	+	-

In the above, $\underline{e}_i^\delta = (q\frac{(1+(n-q)QQ')}{n} - 1)e^\delta(M) + e_i^\delta$, $\overline{e}_i^\delta = \frac{((n-q)qQ'+1)e^\delta(M)-e_i^\delta}{(n-1-\delta(q-1))}$, and all the functions are continuous at these values.

To verify that indeed $L = \emptyset$ for investment profiles as e_i with $i \in M^\delta$ changes, it suffices to check whether $\rho(\overline{e}_i^\delta, e_{-i}^\delta, \delta) < \min e_j^\delta$.

$\rho(\overline{e}_i^\delta, e_{-i}^\delta, \delta) = \frac{(1-\delta)}{(n-\delta q)} < \frac{(1-\delta)}{(n-q)}$.

Observation 4: $\rho(\overline{e}_i^\delta, e_{-i}^\delta, \delta) < \min e_j^\delta$ if $\frac{2(1-\delta)e^\delta(N)}{(n-q)} < \min e_j^\delta$.

When $i \in H^\delta$ deviates,

	$e_i^{\delta 0} \leq e_i < \max\{\widetilde{e}_i^\delta, e_i^{\delta 0}\}$	$\max\{\widetilde{e}_i^\delta, e_i^{\delta 0}\} \leq e_i \leq \widehat{e}_i^\delta$	$\widehat{e}_i^\delta < e_i$
Π	$i \in M, \lvert M\rvert = q$	$i \in M, \lvert M\rvert = q+1$	$i \in H^\delta$
u_i	$V_{M,q}^\delta(e_i, e_{-i}^\delta) = g_1^\delta(e_i)$	$V_{M,q+1}^\delta(e_i, e_{-i}^\delta) = g_2^\delta(e_i)$ $= 1/n$ if $n-q=1$.	$V_{H,n-q}^\delta(e_i, e_{-i}^\delta) = g_3^\delta(e_i)$
u_i'	$+$	$+$ 0 if $n-q=1$	$+$
u_i"	$-$	$-$ 0 if $n-q=1$	$-$
ρ	$\frac{(1-\delta)e(M)e(N)}{(1-\delta)qe(N)+\delta e(H)}$	$\frac{(1-\delta)e(M)e(N)}{(1+q(1-\delta))e(M)+(1+Q)e(H)}$	$\frac{(1-\delta)e(M)e(N)}{(1-\delta)qe(M)+Qe(H)}$
ρ'	+	+	-

In the above, $\widehat{e}_i^\delta = \frac{e^\delta(M)}{Q}$, $\widetilde{e}_i^\delta = (Q+1)e_q^\delta - e^\delta(M)$ and all the functions are continuous at these values.

To check whether $L = \emptyset$ when e_i with $i \in H^\delta$ changes, it suffices to check whether $\rho(\widehat{e}_i^\delta, e_{-i}, \delta) < \min e_j^\delta$, and indeed we see that

$\rho(\widehat{e}_i^\delta, e_{-i}, \delta) = \frac{(1-\delta)e(N)}{(1-\delta)q+Q(Q'(n-q-1)q+\frac{1}{Q})}$

$= \frac{(1-\delta)(Q'(n-q-1)q+\frac{1}{Q})e^\delta(M)}{(1-\delta)q+Q(Q'(n-q-1)q+\frac{1}{Q})} < (1-\delta)e^\delta(M) < \min e_j^\delta$. We have:

Observation 5: If $(1-\delta)e^\delta(M) < \min e_j^\delta$,then $\rho(\widehat{e}_i^\delta, e_{-i}, \delta) < \min e_j^\delta$.

Given the explicit form of the payoff functions, we can check the best response properties directly.

Lemma 4: From $(f_{1i}^\delta)_{i \in M^\delta}$,we see that $f_{1i}^\delta(0) < U_i^\delta$ and so there exists δ' such that $U_i^\delta > u_i^\delta(e_i^{\delta 0}, e_{-i}^\delta|\delta)$ for all $i \in M^\delta$ and $\delta > \delta'$. Also for $\delta < 1$, there is a lower bound on e_i that satisfies this sufficient condition for a best response. □

Let also

$h(\delta, n, q) = U_i^\delta - (g_2^\delta(\widehat{e}_i) - c g_2^{\delta\prime}(\widehat{e}_i))$.

Lemma 5:

$\lim_{\delta \longrightarrow 1} h(\delta, n, q) = \frac{1}{(n-q+1)^2} - \frac{1+q}{(q(n-q)+1)^2} = \frac{(q(n-q)+1)^2 - (1+q)(n-q+1)^2}{(n-q+1)^2(q(n-q)+1)^2} > 0$

Proof: Calculating the numerator, we see

$(q(n-q)+1)^2 - (1+q)(n-q+1)^2 > 0$ for $n \geq 5$, $n-q \geq 2$, and $q > \frac{n}{2}$.□

A.3. Best Response Properties

The candidate equilibria e^δ are stated with constraints on the range of $e_i \in M^\delta$:

$$\begin{cases} e^\delta(M^\delta) = \frac{Q(n-q)}{q[Q(n-q)+1]^2 c} & \\ \text{where } e^\delta(M^\delta) = \sum_{i \in M^\delta} e_i^\delta & i \in M(p^\delta) = M^\delta \\ \text{with } e_i^\delta > U_i^{-1}(u_i(e_i^{\delta 0}|\delta)) & |M^\delta| = q \\ \text{and with } e_i^\delta > 2(1-\delta)\frac{e^\delta(N)}{(n-q)} & \\ e_i^\delta = Q' q e^\delta(M^\delta) & i \in H(p^\delta) = H^\delta, |H^\delta| = n-q \end{cases} \tag{18}$$

and our main claim is

$$\begin{cases} \text{Given } n \text{ and } q, \text{ choose } \delta' \text{ such that} \\ \text{for all } \delta > \delta' \\ h(\delta, n, q) > 0 \\ u_i(e_i^{\delta 0}, e_{-i}^\delta|\delta) < U_i^\delta \quad \text{for all } i \\ \rho(e^\delta, \delta) < 2(1-\delta)\frac{(n-q)q}{[(n-q)+1]^2 c} \\ \text{Then for } n \geq 4,\ e^\delta \text{ is an equilibrium of the reduced game.} \\ \text{If } n = 3, \text{ then } e^\delta \text{ as above with } e_1^\delta/e_2^\delta < 1/5, \text{ is an equilibrium} \\ \text{(where the names of the players are as in Example 1).} \end{cases} \tag{19}$$

To prove the main claim, we need to check that the candidate equilibria satisfy the best response properties. First, we check that it is not worthwhile for players in M^δ, to invest more in Subclaim 1. Subclaim 1 shows that the marginal increase in the bargaining payoff is less than the investment cost in the region where the investment level is so high that the player is the only H player. (Figure 2) We show the similar relation for an M player by Subclaims

2 through 4. By Subclaim 2, we check that when $n-q>1$, reductions in the investment level for a player in H^δ (provided that $|M|=q+1$) by an estimate of the payoff with no investment where we use Lemma 4. (Figure 3) By Subclaim 4, reductions in the investment level for a player in H^δ (hence this player becomes an M player) induce one more player to become an H player and verify that this case does not improve the payoff of the deviating player. (Figure 3) By Subclaim 5, we check that if $n-q=1$, reductions in investment level induce $M^\delta=N$, and as far as $n\geq 4$, the equilibrium net payoff is not worse than $1/n$ implying that this deviation is not worthwhile. Since this does not hold when $n=3$, we must have some restriction on the ratio between e_1^δ and e_2^δ. Finally, Subclaim 5 confirms that there is no best response among $e_i<e_i^{\delta 0}$.

We represent the limit value as δ tends to 1 by the superscript $\delta=1$.

Subclaim 1: e_i with $e_i\geq \overline{e}_i^\delta$ for $i\in M^\delta$ are not best responses.

Proof: We show that $f_{3i}^{\delta\prime}((e^\delta(N)-e_i^\delta)/[\delta(1-q)+n-1])<c$. Then since $(e^\delta(N)-e_i^\delta)/[\delta(1-q)+n-1]\leq \overline{e}_i^\delta$ and f_{3i}^δ is strictly concave, we can conclude that there is no best response among those investment levels with $e_i\geq\overline{e}_i^\delta$.

Recall that $f_{3i}^{\delta\prime}((e^\delta(N)-e_i^\delta)/[\delta(1-q)+n-1])=\frac{(n-1-\delta(q-1))(n-1)}{n^2(e^\delta(N)-e_i^\delta)}<\frac{(n-q+1)(n-1)}{n^2e^\delta(H)}$ and $c=\frac{Q'Q(n-q)}{q\{1+Q'Q(n-q)\}^2e^\delta(M)}=\frac{Q'^2Q(n-q)^2}{\{1+Q'Q(n-q)\}^2e^\delta(H)}>\frac{(n-q)^2}{\{1+(n-q)\}^2e^\delta(H)}$. Since $\frac{n-q}{(n-q+1)^2}\geq\frac{n-1}{n^2}$, we have the desired inequality.

For $Q>1$, we also see that $(n-q)q\geq n-1$ (since $n(q-1)\geq q^2-1$). As for the denominators, $(\delta(1-q)+n-1)>Q$, and $n-1\geq q$, as well, so the claim holds.□

Subclaim 2: For $i\in H^\delta$, $g_2(\widehat{e}_i^\delta)-g_2'(\widehat{e}_i^\delta)\widehat{e}_i^\delta<\frac{1}{\{1+n-q\}^2}$ when $n-q>1$.

Proof: The left-hand-side is the abscissa of the upper support for $g_2^\delta(e_i,e_{-i}^\delta)$ at $\widehat{e}_i^\delta$, and the right-hand-side is less than or equal to U_i^δ. By Lemma 5, we have $h(\delta,n,q)>0$.□

Subclaim 3: For $i\in H^\delta$ when $n-q>1$, $g_2^\delta(e_i)>g_1^\delta(e_i)$ for $e_i<\widetilde{e}_i^\delta$ provided that $\widetilde{e}_i^\delta>0$.

Proof: When $\widetilde{e}_i^\delta=(Q+1)e_q^\delta-e^\delta(M)>0$, for $e_i<\widetilde{e}_i^\delta$, $g_2^\delta(e_i)-g_1^\delta(e_i)=\frac{e^\delta(M)+e_i}{A}-\frac{e_-^\delta+e_i}{A'}$ where $e_-^\delta=e^\delta(M)-e_q^\delta$, $z=Q'(n-q-1)$, $A=(q+1)(e^\delta(M)+e_i)+(Q+1)(zqe^\delta(M))$, $A'=q(e_-^\delta+e_i)+Qzqe^\delta(M)+e_q^\delta)$ and the numerator becomes

$((zq+1)e^\delta(M)+e_i)((Q+1)e_q^\delta-e^\delta(M)-e_i)>0.$ □

Subclaim 4: For $i\in H^\delta$,if $n-q\geq 2$, and $q\geq 3$, $e_i\in[e_i^{\delta 0},\widehat{e}_i^\delta]$ is not a best response.

Proof: By the strict concavity of g_3^δ and the first order condition, $\widehat{e}_i^\delta$ is not a best response. Then Subclaims 2 and 3 show that the subclaim holds.

Subclaim 5: When $n-q=1$, let $\{n\}=H^\delta$. Then e_n^δ is the best response to e_{-n}^δ.

Proof: Since $\widehat{e}_n^\delta \leq e_n, g_3^\delta(e_i)$ is strictly concave and increasing, e_n^δ is the only investment level that satisfies the first order condition. Next observe that when $n>3$, for $e_n<\widehat{e}_n^\delta, u_n(e_n, e_{-n}^\delta|\delta) \leq 1/n$, whereas $U_n^\delta = \frac{Q[(1-\delta)(2q-n)+1]}{\{Q(n-q)+1\}^2} = \frac{(Q)^2}{\{Q+1\}^2} > 1/4 \geq 1/n$. By the monotonicity of u_i, we conclude that $e_n < \widehat{e}_n^\delta$ cannot be a best response.

When $n=3$, let $e_2^\delta \geq e_1^\delta$. $U_3^\delta = \frac{2-\delta}{3-\delta}$, $V_{M,3}(e_1)=1/3, \widehat{e}_3^\delta = (2-\delta)e_2^\delta - e_1^\delta$ and $U_3^\delta > (1/3 - c\widehat{e}_3^\delta)$ holds if $\frac{e_1^\delta}{e_2^\delta} < \frac{7\delta^2-24\delta+18}{-4\delta^2+9\delta}$ ($> \frac{1}{5}$ for $\delta<1$) which holds for $\delta<1$, with $\frac{1}{5} > \frac{e_1}{e_2}$.□

Subclaim 6: There is no best response for those e_i with $e_i < e_i^{\delta 0}$.

Proof: Recall that for those e_i with $e_i < e_i^{\delta 0}$, by the choice of δ, we have $u_i(e_i, e_{-i}|\delta) - ce_i < U_i^\delta$ which implies that there is no best response for those e_i with $e_i < e_i^{\delta 0}$.□

This completes the verification of the best response properties and thus the proof of the proposition is complete.

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